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Non-Time-Orientable Lorentzian Cobordism Allows for Pair Creation

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It is shown that if time-orientability is sacrificed, then, in any spacetime of odd dimension, two cobordant manifolds are also related by a Lorentzian cobordism. Thus, pair creation of Kaluza-Klein monopoles *would* be allowed in such a framework. The even-dimensional case is also examined.

Despite the evident importance of the question, it is still not known whether dynamical topology is a necessary (Sorkin, 1986a) or even a possible (de Witt, 1984) feature of "quantum gravity." Nonetheless, some thought has gone into possible mechanisms by which spatial topology might change, among which the most studied have probably been those in which an n-dimensional manifold interpolates between a pair of spacelike hypersurfaces of possibly distinct topology. Such an interpolating manifold M is called a "cobordism" and various special cases are referred to as Lorentzian, Riemannian, etc., according as M is provided with a Lorentzian metric, a positive-definite metric, etc.

Within the class of Lorentzian cobordisms, one may distinguish between those that admit a consistent choice of future direction and those that do not. The former, time-orientable, ones seem more physically natural, but perhaps not conclusively so, for several reasons. In the first place, the so-far unchallenged *CPT* symmetry of microscopic laws (Cronin, 1981) suggests that spacetime may *not* in fact possess any intrinsic time sense. If it does not, then there is no experimental basis for requiring T-orientability other than reasons stemming from causality itself. But since globally regular Lorentzian cobordisms are in any case causally anomalous (Geroch, 1967), such reasons might seem to rule out (regular) Lorentzian metrics altogether,

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thereby rendering moot the question of whether or not such metrics should be restricted to be T-orientable. Thus, it seems of interest to examine how the inclusion of non-time-orientable metrics would affect the selection rules that obtain in the T-orientable case.

These selection rules are present in addition to the purely (differential) topological obstructions which can block interpolation between two manifolds M_0 and M_1. Assuming that an interpolating manifold M does exist (namely a compact manifold whose boundary is the disjoint union of M_0 and M_1), the further obstruction to a T-oriented metric on M which renders M_0 initial and M_1 final can be deduced (Reinhart, 1963) from the formulas (Sorkin, 1986b)

$$\begin{aligned} \text{ind}(v) &= \chi(M) - \chi(M_0) \\ \text{ind}(-v) &= \chi(M) - \chi(M_1) \end{aligned} \tag{1}$$

Here χ is the Euler number and $\text{ind}(v)$ is an integer which must vanish if M is to admit a metric of the sort in question. In even spacetime dimensions (1) implies that

$$\text{ind}(v) = \chi(M) \tag{2}$$

yielding the condition $\chi(M) = 0$, in order that $\text{ind}(v) = 0$. In odd dimensions (1) implies that

$$\begin{aligned} \text{ind}(v) &= [\chi(M_1) - \chi(M_0)]/2 \\ &=: \Delta\chi/2 \end{aligned} \tag{3}$$

leading to the condition $\Delta\chi = 0$, which is immediately interpretable as a selection rule, inasmuch as $\Delta\chi$ depends only on M_0 and M_1 but not M.

Unfortunately, this selection rule forbids some processes that physical consistency would appear to demand (Witten, 1985), such as monopole pair creation in the $U(1)$ Kaluza–Klein theory (Sorkin, 1986b). The question thus arises whether one could circumvent (3) by allowing non-time-orientable metrics. In particular, we may ask whether such a metric exists in the specific 5-manifold M that presents itself as the natural candidate to mediate creation (or destruction) of a monopole–antimonopole pair.[2]

For this specific M the answer turns out to be "no". Because $\pi_1(M) = 0$, there is in M essentially no distinction between the T-oriented and non-T-oriented cases: every Lorentzian metric on M is T-orientable. The only new freedom therefore would be to use metrics with respect to which both

[2]Such a natural candidate exists for pair creation of any topological particle. (J. L. Friedman, unpublished remark).

M_0 and M_1 were (say) *initial* boundaries. However, such metrics would still be forbidden by the $\Delta\chi = 0$ rule applied now to the transition $\varnothing \to M_1 \amalg M_2$ ($\amalg$ = disjoint union, $\varnothing$ = empty set).

Nevertheless, the following proposition shows that—at the *further* cost of modifying M topologically—we can in fact overcome (3), and indeed similarly convert *any* topological cobordism into one admitting a Lorentzian metric.

Proposition. Let M be a compact (smooth) manifold of odd dimension such that $\partial M = M_0 \amalg M_1$. Then there exists a connected manifold N with the same boundary and with a metric making N into a Lorentzian cobordism between M_0 and M_1.

Proof. If M is not connected, we can make it so by forming the connected sum, $\#$, of its components. Then let v be a Morse vectorfield on M (a vectorfield with isolated zeros which is inward [outward] normal on M_0 [M_1]). By modifying v as explained in Milnor and Weaver (1965, p. 40), we can arrange that its zeros are all nondegenerate (a nondegenerate zero is one at which $\partial_a v_b$ is invertible). According to Lemma 4 of Chapter 6 of Milnor and Weaver (1965), the index of v at any zero will then be ± 1. Let $x \in M$ be a zero of v. If the index there is $+1$, then by the proof of the same Lemma 4 we can arrange that v is purely radial within some disk containing x. If the index at x is -1, then we can do the same for $-v$, which has index $+1$ because $\dim(M)$ is odd. We can therefore assume that each zero of v is contained in a small disk (ball) within which v has the appearance of a purely radial vectorfield (inwardly or outwardly directed).

Let D be such a disk. We would like to remove it and replace it by a manifold K to which v could be extended without developing any zeros. This we know is impossible, but fortunately it suffices to extend not v itself, but only the line-element field $[v]$ defined by v. (A line element is an equivalence class of nonzero vectors, where $v_1 \sim v_2$ iff $v_2 = \pm\lambda v_1$ for some $\lambda > 0$.) From such a field one can form a, not necessarily time-orientable, Lorentzian metric (Steenrod, 1951).

Obviously our extension problem reduces to finding (for any odd n) an n-manifold K with line-element field l, such that $\partial K = S^{n-1}$ and l is normal to ∂K. To that end let $\bar{K}$ be the "cylinder" $D^1 \times S^{n-1}$ (where D^1 is a compact interval) and let $\bar{l}$ be the line-element field parallel to the cylinder axis. Concretely we may embed $\bar{K}$ in R^{n+1} as

$$\{(x_0, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 = 1, \quad -1 \le x_0 \le 1\}$$

and then take for $\bar{l}$ the constant field $[(1, 0, 0, \ldots, 0)]$. Finally, let $K = \bar{K}/Z_2$ be the quotient of $\bar{K}$ by the inversion $x \to -x$. It is clear that $\bar{l}$ induces a well-defined line-element field l on K, that $\partial K \simeq S^{n-1}$, and that l is transverse to ∂K, as needed. ■

Remark. The proof shows that it suffices to glue in ind(v) copies of K. This is equivalent to forming the connected sum of M with as many copies of RP^n.

Now the application of this result to pair creation of Kaluza–Klein monopoles is not entirely straightforward because only cobordisms possessing an asymptotically flat Lorentzian metric are then physically relevant. (An exception might be pair creation in the very early Universe.) However, it should always be possible to carry out the proof in such a way that the original field v looks like a "timelike unit vector" in some neighborhood of spatial infinity, which is then left untouched by the subsequent modifications of M and $[v]$.

Taking this for granted, we learn from the Proposition that the natural pair-creation manifold M described in Sorkin (1986b) can be converted into a substratum for a Lorentzian cobordism by gluing in a single copy of $(S^4 \times D^1)/Z_2$, or equivalently, by forming $M \# RP^5$. [Recall that $\Delta\chi = 2$ for the natural ${}^5M$ and refer to (3).]

In the same way, the vacuum decay found to be forbidden in the context of Sorkin (1986b) would also become possible, though not via the specific cobordism considered in Witten (1982). Hence the instanton estimate of the decay rate made there would not seem to be reliable within the present, Lorentzian framework.

In even spacetime dimensions no new possibilities seem to be opened up by allowing non-time-oriented Lorentzian cobordisms. In fact, by gluing such a cobordism M to itself and passing, if necessary, to the time-oriented double covering of the resulting closed spacetime, one finds that $\chi(M) = 0$. But then M (if connected) actually admits a *T-oriented* cobordism, in view of (1) and the fact that $\chi(M_0) = 0$, since $\dim(M_0)$ is odd.

In particular, the two-dimensional transition $S^1 \amalg S^1 \to S^1$ is still impossible, even in the non-T-oriented case: neither the "trousers" nor any other (connected) mediating manifold will admit a globally Lorentzian metric (rendering the boundary spacelike). On the other hand, the demand for a T-oriented metric poses no obstacle at all in even dimensions greater than 3, as follows from Theorem (1) of Reinhart (1963). [Warning: Lemma (2) there is wrong for $n = 1$; but this does not affect the proof for dimension ≥ 3.] The special character of $n = 2$ revealed here suggests that the example of topology change studied by de Witt (1984) may be misleading as a guide to what would occur in dimensions 3 or higher.

To summarize, then, let M_0 and M_1 be closed manifolds of dimension $n - 1$, and let M be an n-manifold whose boundary comprises M_0 and M_1.

For $n \geq 3$ there always exists a second, connected manifold N and a Lorentzian metric thereon with respect to which both M_0 and M_1 are spacelike. For n even (and ≥ 4) the metric can always be chosen with a time orientation rendering M_0 initial and M_1 final. For n odd such a T-oriented metric exists if and only if $\chi(M_0) = \chi(M_1)$.

The present paper has concentrated upon the possibility that the apparently natural requirement of time-oriented metrics might be dropped. It is also possible, as suggested by CP violation, that a requirement of overall spacetime orientation should be added, or even that temporal and spatial orientations should both be imposed.[3] But it seems equally possible that topology change will be understood, not by weakening the requirements of causality, as we have done here, but by upholding them even above the apparently more fundamental mathematical notion of spacetime as a differentiable manifold with everywhere invertible metric.

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[3]See Theorem (2) of Reinhart (1963) concerning a certain case of oriented cobordism.